

408348

CATALOGED BY DDC

AS AD NO.

408348



63-4-2

DETERMINATION OF PARAMETERS FOR CORRELATED DATA  
BY THE USE OF A GENERALIZED LEAST-SQUARES  
CRITERION INVOLVING LINEARIZED RESIDUALS

NAVWEPS REPORT 7942  
NOTS TP 2979  
COPY 12

by

Otto Neall Strand  
Aviation Ordnance Department

Released to ASTIA for further dissemination without limitations beyond those imposed by security regulations.

**ABSTRACT.** This report extends the least-squares methods currently in use at NOTS to cover the case of correlated data. A derivation of the theory is followed by detailed discussions of the applications to the Askania cinetheodolite solution and curve fitting of space-position data. References are given for other local applications, but specific results for these are not included.

DDC  
RECEIVED  
JUL 10 1963  
RECORDED  
TICK B

U. S. NAVAL ORDNANCE TEST STATION

China Lake, California  
April 1963

**U. S. NAVAL ORDNANCE TEST STATION**  
**AN ACTIVITY OF THE BUREAU OF NAVAL WEAPONS**

## FOREWORD

The study described in this report extends mathematical methods to cover cases that are of specific interest at the U. S. Naval Ordnance Test Station. The derivation of the theory is followed by its application to the analysis of data that must be obtained in the evaluation of weapon systems.

This study was made early in 1961 under departmental overhead funds. The report has been reviewed for technical accuracy by D. E. Zilmer and A. J. Rice.

Released by  
A. G. HOYEM, Head,  
Aircraft Projects Div.  
12 June 1962

Under authority of  
N. E. WARD, Head,  
Aviation Ordnance Dept.

**NOTS Technical Publication 2979  
NAVWEPS Report 7942**

Published by ..... Publishing Division  
Technical Information Department  
Supersedes ..... IDP 1348  
Collation ..... Cover, 5 leaves, abstract cards  
First printing ..... 80 numbered copies  
Security classification ..... UNCLASSIFIED

## INTRODUCTION

This report extends certain least-squares methods currently in use at the Naval Ordnance Test Station (NOTS) to cover the case of correlated data. This extension makes possible, for instance, the use of derived azimuths and elevations in an Askania solution for space position. The general theory of least squares is given in the literature (Ref. 1-3). This report contains independent derivations pertaining to certain cases of special interest at NOTS. A presentation of the theory is followed by detailed discussions of the applications to the Askania cinetheodolite solution and curve fitting of space-position data. References to other local applications are given, but the specific results for these are not presented.

## DERIVATION OF THE THEORY

### THE COVARIANCE MATRIX OF A LINEAR COMBINATION OF RANDOM VARIABLES

**THEOREM 1.** *Suppose*

$$U = DV \text{ where}$$

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix}, \quad V = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}, \quad D = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{k1} & d_{k2} & \dots & d_{kn} \end{pmatrix}$$

the  $X_i$  are random variables, and the  $d_{ij}$  are constants. Further define

$$S = \begin{pmatrix} \sigma_{X_1}^2 & \sigma_{X_1 X_2} & \dots & \sigma_{X_1 X_n} \\ \sigma_{X_1 X_2} & \sigma_{X_2}^2 & \dots & \sigma_{X_2 X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{X_1 X_n} & \sigma_{X_2 X_n} & \dots & \sigma_{X_n}^2 \end{pmatrix}$$

and

$$\bar{\sigma} = \begin{pmatrix} \sigma_{u_1}^2 & \sigma_{u_1 u_2} & \dots & \sigma_{u_1 u_k} \\ \sigma_{u_1 u_2} & \sigma_{u_2}^2 & \dots & \sigma_{u_2 u_k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{u_1 u_k} & \dots & \sigma_{u_k}^2 \end{pmatrix}$$

Here  $\sigma_{uv} = \text{cov}(u, v)$ . Then  $\bar{\sigma} = DSD^T$ .

*Proof.* If  $X_1, X_2, \dots, X_n$  are random variables and  $a_i, b_i$  are constants with

$$T_1 = a_1 X_1 + a_2 X_2 + \dots + a_n X_n$$

$$T_2 = b_1 X_1 + b_2 X_2 + \dots + b_n X_n$$

then by taking expected values we obtain

$$E(T_1 T_2) = E\left[\left(\sum_{i=1}^n a_i X_i\right)\left(\sum_{j=1}^n b_j X_j\right)\right] = E\left(\sum_{i,j=1}^n a_i b_j X_i X_j\right) = \sum_{i,j=1}^n a_i b_j E(X_i X_j) \quad (1)$$

$$E(T_1)E(T_2) = E\left(\sum_{i=1}^n a_i X_i\right)E\left(\sum_{j=1}^n b_j X_j\right) = \sum_{i=1}^n a_i E(X_i) \sum_{j=1}^n b_j E(X_j) = \sum_{i,j=1}^n a_i b_j E(X_i)E(X_j) \quad (2)$$

In obtaining Eq. 1 and 2, the linearity of the expected-value operator has been used and the products of sums have been written as double sums. Noting that by definition  $\sigma_{uv} = E(uv) - E(u)E(v)$ , the subtraction of Eq. 2 from Eq. 1 gives

$$\sigma_{T_1 T_2} = \sum_{i,j=1}^n a_i b_j \sigma_{X_i X_j} \quad (3)$$

and putting  $a_i = d_{li}$  and  $b_i = d_{pj}$  so that

$$T_1 = u_l = d_{l1} X_1 + d_{l2} X_2 + \dots + d_{ln} X_n$$

$$T_2 = u_p = d_{p1} X_1 + d_{p2} X_2 + \dots + d_{pn} X_n$$

gives, by virtue of Eq. 3,

$$\sigma_{u_l u_p} = \sum_{i,j=1}^n d_{li} d_{pj} \sigma_{X_i X_j} \quad (4)$$

From direct calculation of the  $lp$  element of  $DSD^T$ , denoted by  $(DSD^T)_{lp}$ , one obtains

$$(DSD^T)_{lp} = \sum_{i,j=1}^n d_{li} d_{pj} \sigma_{X_i X_j} \quad (5)$$

A comparison of Eq. 4 and 5 gives

$$DSD^T = (\sigma_{u_l u_p})$$

as required.

## A FORMULA FOR THE DIFFERENTIATION OF A QUADRATIC FORM

The result obtained here is well known, but is established in a convenient form for use in deriving the normal equations in the following section.

THEOREM 2. *Let*

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

$$P = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1m} \\ p_{12} & p_{22} & \dots & p_{2m} \\ \dots & \dots & \dots & \dots \\ p_{1m} & p_{2m} & \dots & p_{mm} \end{pmatrix}, \quad L = \begin{pmatrix} l_{11} & l_{12} & \dots & l_{1n} \\ l_{21} & l_{22} & \dots & l_{2n} \\ \dots & \dots & \dots & \dots \\ l_{m1} & l_{m2} & \dots & l_{mn} \end{pmatrix}$$

$$F(U) = (LU + C)^T P (LU + C)$$

$$\frac{dF}{dU} = \begin{pmatrix} \frac{\partial F}{\partial u_1} \\ \frac{\partial F}{\partial u_2} \\ \vdots \\ \frac{\partial F}{\partial u_n} \end{pmatrix}$$

Then

$$\frac{dF}{dU} = 2L^T P(LU + C)$$

*Proof.* It can be verified by expansion and differentiation that

$$\frac{d}{dW} (W^T BW) = 2BW$$

and  $W$  is a column vector; also

$$\frac{d}{dW} KW = K^T$$

if  $K$  is a row vector.

By the Distributive Law

$$F(U) = (C^T + U^T L^T)P(LU + C) = C^T PC + C^T PLU + U^T L^T PLU + U^T L^T PC$$

Since  $C^T PLU$  is a  $1 \times 1$  matrix and as such is symmetric, it follows that

$$C^T PLU = U^T L^T P^T C = U^T L^T PC$$

Thus,

$$F(U) = C^T PC + 2C^T PLU + U^T (L^T PL)U$$

Hence, by the formulas already derived,

$$\frac{dF}{dU} = 2L^T PLU + 2(C^T PL)^T = 2L^T P(LU + C)$$

as required.

### THE GENERALIZED LEAST-SQUARES CRITERION: DERIVATION AND DISCUSSION OF NORMAL EQUATIONS

Let it be required to determine the parameters  $u_1, u_2, \dots, u_k$  from measurements  $m_1, m_2, \dots, m_n$  with covariance matrix  $M$ . Define

$$V = \begin{pmatrix} m_1 - a_1 \\ m_2 - a_2 \\ \dots \\ m_n - a_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \quad U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix}$$

$$Q = \begin{pmatrix} q_{11} & q_{21} & \dots & q_{k1} \\ q_{12} & q_{22} & \dots & q_{k2} \\ \dots & \dots & \dots & \dots \\ q_{1n} & q_{2n} & \dots & q_{kn} \end{pmatrix}, \quad M = \begin{pmatrix} \sigma_{v_1}^2 & \sigma_{v_1 v_2} & \dots & \sigma_{v_1 v_n} \\ \sigma_{v_1 v_2} & \sigma_{v_2}^2 & \dots & \sigma_{v_2 v_n} \\ \dots & \dots & \dots & \dots \\ \sigma_{v_1 v_n} & \sigma_{v_2 v_n} & \dots & \sigma_{v_n}^2 \end{pmatrix}$$

Here the  $a_i$ ,  $q_{ij}$  and  $\sigma_{v_i v_j}$  are constants,  $M$  is positive definite, and  $Q$  is of rank  $k$ . The latter assumption implies that there are sufficient independent data to solve the problem.

It can be shown (Ref. 2 and 3) that the maximum-likelihood estimate of  $u_1, u_2, \dots, u_k$  under the assumption of normally distributed errors in the  $m_i$  is given by

$$G(U) = (V - QU)^T M^{-1} (V - QU) = \text{minimum} \quad (6)$$

This criterion is taken as the generalized least-squares criterion for correlated data.

By Theorem 2,

$$\frac{dG}{dU} = 2(-Q)^T M^{-1} (V - QU)$$

Equating to zero gives the normal equations

$$AU = Q^T M^{-1} V \quad (7)$$

where

$$A = Q^T M^{-1} Q$$

We show that  $A$  is non-singular. Consider the quadratic form  $H(X) = X^T M^{-1} X$ , which is positive definite. If we perform a linear transformation,  $X = QY$ , there results

$$H(QY) = Y^T (Q^T M^{-1} Q) Y$$

By a theorem (Ref. 4) from linear algebra,

$$\text{rank } Q + \text{nullity } Q = \text{number of columns of } Q$$

Since  $\text{rank } Q = \text{number of columns of } Q = k$ , it follows that  $\text{nullity } Q = 0$ . Thus  $QY$  can be zero only if  $Y = 0$ . Hence  $H(QY)$ , when regarded as a quadratic form in  $Y$ , is positive definite;  $Q^T M^{-1} Q$  is a positive-definite matrix and, as such, has only positive eigenvalues. That is to say,  $Q^T M^{-1} Q$  is non-singular. (Note that if  $Q$  had rank less than  $k$ , then  $A$  would be singular, since the rank of a product is at most equal to the rank of any factor.)

Equation 7 can therefore be solved uniquely for  $U$  as follows:

$$U = A^{-1} Q^T M^{-1} V \quad (8)$$

In order to prove that Eq. 8 actually furnishes a minimum of the expression for  $G(U)$ , Eq. 6, let  $U_0$  be the solution given by Eq. 8. That is, let  $U_0 = A^{-1} Q^T M^{-1} V$  and let  $U^*$  be any other real column vector having dimension  $k$ . It follows by direct expansion, making use of this expression for  $U_0$ , that

$$G(U^*) - G(U_0) = (U^* - U_0)^T A (U^* - U_0)$$

Since it has already been shown that  $A$  is positive definite, it is concluded that  $U_0$  does indeed furnish a minimum of Eq. 6.

If  $S_U$  is the variance-covariance matrix of  $U$ , we have by a direct application of Theorem 1

$$S_U = (A^{-1} Q^T M^{-1}) M (A^{-1} Q^T M^{-1})^T = A^{-1} (Q^T M^{-1} Q) A^{-1}$$

or

$$S_U = A^{-1} \quad (9)$$

Thus the entire least-squares solution consists of computing  $S_U$  by Eq. 9 and  $U$  by Eq. 8. The application of this solution to the various specific situations merely consists of properly defining the matrix  $Q$  and the quantities  $a_i$ . This is done in detail for the Askania cinetheodolite solution and the application to curve fitting of space-position data. References are given for other applications.

## APPLICATIONS

### APPLICATION TO THE ASKANIA CINETHEODOLITE SOLUTION

For a more detailed explanation of the notation used in this section the reader may consult Ref. 5. Let  $r$  be the number of Askania stations. Then,  $n = 2r$ . It is assumed that the  $i$ th Askania station determines azimuth and elevation measurements  $A_i$  and  $E_i$  with covariance matrix

$$\begin{pmatrix} \sigma_{A_i}^2 & \sigma_{A_i E_i} \\ \sigma_{A_i E_i} & \sigma_{E_i}^2 \end{pmatrix} \quad (i = 1, 2, \dots, r)$$

and that the measurement errors from a given station are independent of those from any other station. Of course, since  $A_i$  and  $E_i$  are assumed correlated here, it would be permissible to use azimuths and elevations derived from other primary measurements rather than direct Askania readings. Under these conditions,

$$M = \left( \begin{array}{cccc|cccc} \sigma_{A_1}^2 & 0 & \cdot & 0 & \sigma_{A_1 E_1} & 0 & 0 & 0 \\ 0 & \sigma_{A_2}^2 & 0 & 0 & 0 & \sigma_{A_2 E_2} & 0 & 0 \\ \cdot & \cdot \\ 0 & 0 & \cdot & \sigma_{A_r}^2 & 0 & 0 & \cdot & \sigma_{A_r E_r} \\ \hline \sigma_{A_1 E_1} & 0 & \cdot & 0 & \sigma_{E_1}^2 & 0 & \cdot & 0 \\ 0 & \sigma_{A_2 E_2} & 0 & 0 & 0 & \sigma_{E_2}^2 & \cdot & 0 \\ \cdot & \cdot \\ 0 & 0 & \cdot & \sigma_{A_r E_r} & 0 & 0 & \cdot & \sigma_{E_r}^2 \end{array} \right)$$

We have

$$m_i = i\text{th azimuth reading, } i = 1, 2, \dots, r$$

$$m_i = (i - r)\text{th elevation reading, } i = r + 1, \dots, 2r$$

In order to obtain  $M^{-1}$  as required for the normal equations, one may follow the analysis<sup>1</sup> of Ref. 6 with the following result, which is easily verified by direct calculation.

<sup>1</sup> The use of this procedure, which finds  $M^{-1}$  in terms of submatrices that are diagonal, was suggested by Mrs. D. Saitz of the Test Department, NOTS.

$$M^{-1} = \begin{vmatrix} \frac{\sigma_{E_1}^2}{D_1} & 0 & \cdot & 0 & \frac{-\sigma_{A_1 E_1}}{D_1} & 0 & \cdot & 0 \\ 0 & \frac{\sigma_{E_2}^2}{D_2} & 0 & 0 & 0 & \frac{-\sigma_{A_2 E_2}}{D_2} & 0 & 0 \\ \cdot & \cdot \\ 0 & 0 & \cdot & \frac{\sigma_{E_r}^2}{D_r} & 0 & \cdot & \cdot & \frac{-\sigma_{A_r E_r}}{D_r} \\ \hline \frac{-\sigma_{A_1 E_1}}{D_1} & 0 & \cdot & 0 & \frac{\sigma_{A_1}^2}{D_1} & 0 & \cdot & 0 \\ 0 & \frac{-\sigma_{A_2 E_2}}{D_2} & 0 & 0 & 0 & \frac{\sigma_{A_2}^2}{D_2} & 0 & 0 \\ \cdot & \cdot \\ 0 & 0 & \cdot & \frac{-\sigma_{A_r E_r}}{D_r} & 0 & \cdot & 0 & \frac{\sigma_{A_r}^2}{D_r} \end{vmatrix}$$

where  $D_i = \sigma_{A_i}^2 \sigma_{E_i}^2 - (\sigma_{A_i E_i})^2$ . None of the  $D_i$  will be zero, for if  $D_i = 0$ , then the corresponding covariance matrix

$$\begin{pmatrix} \sigma_{A_i}^2 & \sigma_{A_i E_i} \\ \sigma_{A_i E_i} & \sigma_{E_i}^2 \end{pmatrix}$$

would be singular.

The unknowns are the corrections  $u_1 = \Delta x$ ,  $u_2 = \Delta y$ , and  $u_3 = \Delta z$  to be applied to an initial estimate  $x_0$ ,  $y_0$ ,  $z_0$  of space position. Therefore  $k = 3$ . We define the coordinates of the  $i$ th Askania station as  $X_i$ ,  $Y_i$ ,  $Z_i$  where  $i = 1, \dots, r$ .

Then

$$a_i = \tan^{-1} \frac{z_0 - Z_i}{x_0 - X_i} \quad (i = 1, 2, \dots, r)$$

$$a_i = \tan^{-1} \frac{y_0 - Y_{i-r}}{[(x_0 - X_{i-r})^2 + (y_0 - Y_{i-r})^2]^{1/2}} \quad (i = r+1, \dots, n)$$

The  $q_{ji}$  are defined as follows:

$$q_{1i} = \frac{Z_i - z_0}{(X_i - x_0)^2 + (Z_i - z_0)^2} \quad (i = 1, 2, \dots, r)$$

$$q_{1i} = \frac{-(X_{i-r} - x_0)(Y_{i-r} - y_0)}{[(X_{i-r} - x_0)^2 + (Y_{i-r} - y_0)^2 + (Z_{i-r} - z_0)^2][(X_{i-r} - x_0)^2 + (Z_{i-r} - z_0)^2]^{1/2}} \quad (i = r + 1, \dots, n)$$

$$q_{2i} = 0 \quad (i = 1, 2, \dots, r)$$

$$q_{2i} = \frac{[(X_{i-r} - x_0)^2 + (Z_{i-r} - z_0)^2]^{1/2}}{[(X_{i-r} - x_0)^2 + (Y_{i-r} - y_0)^2 + (Z_{i-r} - z_0)^2]} \quad (i = r + 1, \dots, n)$$

$$q_{3i} = \frac{-(X_i - x_0)}{(X_i - x_0)^2 + (Z_i - z_0)^2} \quad (i = 1, 2, \dots, r)$$

$$q_{3i} = \frac{-(Y_{i-r} - y_0)(Z_{i-r} - z_0)}{[(X_{i-r} - x_0)^2 + (Y_{i-r} - y_0)^2 + (Z_{i-r} - z_0)^2][(X_{i-r} - x_0)^2 + (Z_{i-r} - z_0)^2]^{1/2}} \quad (i = r + 1, \dots, n)$$

The solution shown here is ordinarily iterated until all the corrections  $u_i$  are negligible. For further detail see Ref. 5.

### APPLICATION TO CURVE FITTING OF SPACE-POSITION DATA

Suppose that it is required to fit polynomials in the time,  $t$ , to space position,  $x, y, z$ , in the form

$$x = u_1 + u_2 t + \dots + u_{l_1+1} t^{l_1}$$

$$y = u_{l_1+2} + u_{l_1+3} t + \dots + u_{l_1+l_2+2} t^{l_2}$$

$$z = u_{l_1+l_2+2} + u_{l_1+l_2+3} t + \dots + u_{l_1+l_2+l_3+3} t^{l_3}$$

The quantities required to apply the method of this report are obtained below.

The basic data are  $x_i, y_i, z_i, M_i, t_i$  (space positions, variance-covariance matrices, and values of time) for  $i = 1, 2, \dots, r$ , where

$$M_i = \begin{pmatrix} \sigma_{x_i}^2 & \sigma_{x_i y_i} & \sigma_{x_i z_i} \\ \sigma_{x_i y_i} & \sigma_{y_i}^2 & \sigma_{y_i z_i} \\ \sigma_{x_i z_i} & \sigma_{y_i z_i} & \sigma_{z_i}^2 \end{pmatrix}$$

By definition,  $n = 3r$ ,  $a_i = 0$  for  $i = 1, 2, \dots, 3r$ ,

$$\begin{pmatrix} m_1 \\ m_2 \\ \cdot \\ \cdot \\ m_{3r} \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \\ \cdot \\ \cdot \\ x_r \\ y_r \\ z_r \end{pmatrix}$$

$$Q = \left( \begin{array}{cccc|cccc|cccc} 1 & t_1 & \cdot & t_1^{l_1} & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & 0 & 1 & t_1 & \cdot & t_1^{l_2} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 & 1 & t_1 & \cdot & t_1^{l_3} \\ \hline 1 & t_2 & \cdot & t_2^{l_1} & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 1 & t_2 & \cdot & t_2^{l_2} & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & 0 & 1 & t_2 & \cdot & t_2^{l_3} \\ \hline \cdot & \cdot \\ 1 & t_r & \cdot & t_r^{l_1} & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 1 & t_r & \cdot & t_r^{l_2} & 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & 0 & 1 & t_r & \cdot & t_r^{l_3} \end{array} \right)_{3_r}$$

$$M = \left( \begin{array}{ccccc} M_1 & 0 & \cdot & \cdot & 0 \\ 0 & M_2 & 0 & \cdot & 0 \\ 0 & 0 & M_3 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & M_r \end{array} \right)_{3_r}$$

Hence, by inspection,

$$M^{-1} = \begin{vmatrix} M_1^{-1} & & & & \\ & M_2^{-1} & & & \\ & & M_3^{-1} & & 0 \\ & & & \ddots & \\ 0 & & & & M_r^{-1} \end{vmatrix}$$

Then the vector of polynomial-equation coefficients is given by Eq. 8, and the variance-covariance matrix of these coefficients is given by Eq. 9. The smoothed  $x, y, z$  values obtained by evaluation at time  $t$ , together with the corresponding variance-covariance matrix, are easily obtained.<sup>2</sup>

## REFERENCES TO OTHER LOCAL APPLICATIONS

The required quantities for application of the method of this report to several specialized least-squares procedures are easily obtained from Ref. 7 and 8 and several informal reports<sup>3,4,5</sup>. It must be admitted that these solutions would not be affected by the use of the method of this report. This is true because the assumption of uncorrelated data is valid in terms of present knowledge about the measurement techniques involved. However, the possibility does exist that the examples under APPLICATIONS can all be made special cases of the general method. Furthermore, a general least-squares subroutine based on the method of this report could easily be written for the IBM 7090 computer.

## SUMMARY

A least-squares procedure for correlated data has been presented. The use of this method will not affect present solutions unless the assumption of uncorrelated data is to be replaced by an estimate of the variance-covariance matrix. The method of this report is applicable at present to (1) the use of derived azimuths and elevations, which are correlated, to obtain space position; and (2) the fitting of space-position data by polynomials in time where the variance-covariance matrix for each given time is known. It is recommended that a general least-squares subroutine incorporating the equations of this report be programmed.

<sup>2</sup> An informal report, IDP 1339, entitled "The Determination of the Variances and Covariances of Line-of-Sight Angular Rates as Obtained From Askania Data," by Otto Neall Strand, was issued by NOTS 5 December 1961.

<sup>3</sup> Informal report, Technical Note 303-26, entitled "COTAR Data Reduction and Error Analysis," by Otto Neall Strand, was issued by NOTS in September 1957.

<sup>4</sup>IDP 1272, entitled "A Least-Squares Star Calibration of the FIR Camera," by Otto Neall Strand and Lee Thomson, was issued by NOTS on 26 June 1961.

51DP 1313, entitled "A Least-Squares FLR Solution for Aircraft Space Position," by Otto Neall Strand and Lee Thomson, was issued by NOTS on 12 July 1961.

**REFERENCES**

1. Arley, Niels, and K. R. Buch. *Introduction to the Theory of Probability and Statistics*. New York, Wiley, 1950.
2. Ballistic Research Laboratories. *A Matrix Treatment of the General Problem of Least Squares Considering Correlated Observations*, by Duane C. Brown. Aberdeen Proving Ground, Maryland, May 1955. (BRL Report 937.)
3. Scheffé, Henry. *The Analysis of Variance*. New York, Wiley, 1959.
4. Perlis, Sam. *Theory of Matrices*. Cambridge, Mass., Addison-Wesley, 1952. P. 54.
5. U. S. Naval Ordnance Test Station, Inyokern. *Techniques for the Statistical Analysis of Cine-theodolite Data*, by R. C. Davis. China Lake, Calif., NOTS, 22 March 1951. (NAVORD Report 1299, NOTS 369.)
6. Faddeeva, V. N. *Computational Methods of Linear Algebra*, tr. by Curtis D. Benster. New York, Dover, 1959. Pp. 102-03.
7. U. S. Naval Ordnance Test Station. *Mathematical Methods Used To Determine the Position and Attitude of an Aerial Camera*, by Otto Neall Strand. China Lake, Calif., NOTS, March 1956. (NAVORD Report 5333, NOTS 1585.)
8. U. S. Naval Ordnance Test Station, Inyokern. *Estimation of Missile Position From Radar Slant-Range Measurements*, by Olaf E. W. Heimdal. China Lake, Calif., NOTS, 9 April 1951. (NAVORD Report 1305, NOTS 375.)

ABS RA

ARO

U. S. Naval Ordnance Test Station  
*Determination of Parameters for Correlated Data by the Use of a Generalized Least-Squares Criterion Involving Linearized Residuals*,  
by Otto Neall Strand. China Lake, Calif., NOTS, April 1963. 10 pp.  
(NAVWEPS Report 7942, NOTS TP 2979), UNCLASSIFIED.

ABSTRACT. This report extends the least-squares methods currently in use at NOTS to cover the case of correlated data. A derivation of the theory is followed by detailed discussions of the applications to the Askania cinetheodolite solution and curve fitting of space-position data. References are given for other local applications, but specific results for these are not included.

1 card, 4 copies



U. S. Naval Ordnance Test Station  
*Determination of Parameters for Correlated Data by the Use of a Generalized Least-Squares Criterion Involving Linearized Residuals*,  
by Otto Neall Strand. China Lake, Calif., NOTS, April 1963. 10 pp.  
(NAVWEPS Report 7942, NOTS TP 2979), UNCLASSIFIED.

ABSTRACT. This report extends the least-squares methods currently in use at NOTS to cover the case of correlated data. A derivation of the theory is followed by detailed discussions of the applications to the Askania cinetheodolite solution and curve fitting of space-position data. References are given for other local applications, but specific results for these are not included.

1 card, 4 copies



U. S. Naval Ordnance Test Station  
*Determination of Parameters for Correlated Data by the Use of a Generalized Least-Squares Criterion Involving Linearized Residuals*,  
by Otto Neall Strand. China Lake, Calif., NOTS, April 1963. 10 pp.  
(NAVWEPS Report 7942, NOTS TP 2979), UNCLASSIFIED.

ABSTRACT. This report extends the least-squares methods currently in use at NOTS to cover the case of correlated data. A derivation of the theory is followed by detailed discussions of the applications to the Askania cinetheodolite solution and curve fitting of space-position data. References are given for other local applications, but specific results for these are not included.

U. S. Naval Ordnance Test Station  
*Determination of Parameters for Correlated Data by the Use of a Generalized Least-Squares Criterion Involving Linearized Residuals*,  
by Otto Neall Strand. China Lake, Calif., NOTS, April 1963. 10 pp.  
(NAVWEPS Report 7942, NOTS TP 2979), UNCLASSIFIED.

ABSTRACT. This report extends the least-squares methods currently in use at NOTS to cover the case of correlated data. A derivation of the theory is followed by detailed discussions of the applications to the Askania cinetheodolite solution and curve fitting of space-position data. References are given for other local applications, but specific results for these are not included.

## **INITIAL DISTRIBUTION**

**3 Chief, Bureau of Naval Weapons**  
DLI-31 (2)  
R-3 (1)  
**2 Naval Weapons Services Office**  
**10 Armed Services Technical Information Agency (TIPCR)**